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A geometric proof of the Lelong-Poincaré formula

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Abstract

We propose a geometric proof of the fundamental Lelong-Poincaré formula : $dd^c \log|f| = [f = 0]$ where f is any nonzero holomorphic function defined on a complex analytic manifold V and $[f = 0]$ is the integration current on the divisor of the zeroes of f.

Our approach is based, via the local parametrization theorem, on a precise study of the local geometry of the hypersurface given by f. Our proof extends naturally to the meromorphic case.

Keywords : *Complex analytic manifolds, analytic sets, local parametrization theorem; integration currents, branching coverings.*

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Since the Lelong-Poincaré formula plays a crucial role in complex analytic geometry, notably in intersection theory (see [4]), it is a natural aim to look for a geometric proof of this fundamental formula. More precisely, we offer a geometric proof of the following

Theorem *Let V be a connected complex analytic manifold of dimension n and let $f : V \to \mathbf{C}$ be a holomorphic nonzero function. Then, the meromorphic differential form $d'f/f$ defines a current of type (1,0) on V, and furthermore, we have*

$$(LP) \qquad d''\left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right] \;=\; [f=0]$$

where $[f=0]$ is the integration current on the divisor of the zeroes of f.

We denote by $\mathcal{D}(V)$ the set of compactly supported differential forms of class C^∞ in V.

Recall that if T is a current of degree s on V, then dT is the current of degree $s+1$ acting by the rule :

$$dT\,(\varphi) \;=\; \Big\langle dT, \varphi \Big\rangle \;:=\; (-1)^{s+1}\,\Big\langle T, d\varphi \Big\rangle \;=\; (-1)^{s+1}\,T\,(d\varphi), \quad \varphi \in \mathcal{D}(V),$$

with $d = d' + d''$ where d' and d'' are holomorphic and antiholomorphic differentiation operator, respectively. So, for any $(n-1, n-1)$-form φ in $\mathcal{D}(V)$, we have

$$\begin{aligned} d\left[\frac{1}{2i\pi}\,\frac{df}{f}\right](\varphi) = d\left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right](\varphi) \;&=\; \left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right](d\varphi) \\ = \left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right](d''\varphi) \;&=\; d''\left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right](\varphi). \end{aligned}$$

Recall also that the integration current exists on any analytic set (see [6]), and if ω is a locally integrable (p,q)- form on V, it defines a current $[\omega]$ of type (p,q) on V by the formula :

$$[\omega]\,(\varphi) \;:=\; \Big\langle [\omega], \,\varphi \Big\rangle \;=\; \int_V \omega \wedge \varphi,$$

where φ is any $(n-p, n-q)$ - form in $\mathcal{D}(V)$.

Now, here is the outline of our proof.

By an argument of partition of unity, we easily see that our problem is local on V, and then we may assume that V is a domain (open and connected set) of $\mathbf{C}^n$ containing 0, such that $f(0) = 0$ and f is nonzero on V.

The proof of the theorem can be divided into five steps :

– Existence of the current $[d'f/f]$ when $V = \mathbf{C}$ and $f = P \in \mathbf{C}[z]$.

– Proof of (LP) when $V = \mathbf{C}$ and $f = P \in \mathbf{C}[z]$.

– Existence of the current $[d'f/f]$ in the general case.

– Proof of (LP) for a special class of test forms.

– Proof of (LP) in the general case.

The two firs steps are quite elementary applications from analysis in one complex variable. We focus now on the three last steps.

§1. Existence of the current $[d'f/f]$ in the general case.

As f is nonzero on V, we can choose a local coordinates system $(z_1, \ldots, z_n)$ such that, for any $j \in \{1, \ldots, n\}$, the partial function $\xi \mapsto f(0, \ldots, \xi, \ldots, 0)$, obtained by varying z_j, is nonzero in a neighborhood of the origin.

Then we have

$$\frac{d'f(z_1, \ldots, z_n)}{f(z_1, \ldots, z_n)} = \sum_{j=1}^{n} \frac{1}{f(z_1, \ldots, z_n)} \frac{\partial f(z_1, \ldots, z_n)}{\partial z_j} \; dz_j,$$

and we are going to see that the coefficient of dz_n is locally integrable on V. The proof is obviously analogous for the coefficients of dz_j when $1 \leq j \leq n-1$.

By the Weierstrass preparation theorem, and up to a restriction of the open set V, we may assume that $V = \Omega \times D(0,\varepsilon)$, $\varepsilon > 0$, where Ω is a domain in $\mathbf{C}^{n-1}$ containing 0, and where f can be written, by setting $t = (z_1, \ldots, z_{n-1})$ and $z = z_n$:

$$f(t,z) \;=\; I(t,z)\; P_t(z), \quad (t,z) \in \Omega \times D(0,\varepsilon), \tag{1}$$

where I is an analytic function in V with values in $\mathbf{C}^*$, and P_t is a monic polynomial of degree k (k being the multiplicity of the function $\xi \mapsto f(0, \ldots, 0, \xi)$ at 0) which depends analytically on t and whose roots $z^j(t)$ are in $D(0,\varepsilon)$ for any $t \in \Omega$.

Now consider a compact set $K = K_1 \times K_2$ with K_1 and K_2 are compact sets in Ω and $D(0,\varepsilon)$ respectively.

Since

$$\frac{1}{f(t,z)}\;\frac{\partial f(t,z)}{\partial z} \;=\; \frac{1}{I(t,z)}\;\frac{\partial I(t,z)}{\partial z} \;+\; \frac{1}{P_t(z)}\;\frac{\partial P_t(z)}{\partial z}$$

and as the meromorphic form $d'I/I$ has no singularity in the open set $\Omega \times D(0,\varepsilon)$, it is enough to prove that, for each $j \in \{1, \ldots, p\}$, the integral

$$\int_K \frac{1}{|z - z^j(t)|}\;\left(\frac{i}{2}\right)^n\; dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_{n-1} \wedge d\overline{z}_{n-1} \wedge dz \wedge d\overline{z}$$

is finite because

$$\frac{1}{P_t(z)}\;\frac{\partial P_t(z)}{\partial z} \;=\; \sum_{j=1}^{k}\;\frac{1}{z - z^j(t)}.$$

For this purpose, take a number $r \in]\varepsilon/3, 2\varepsilon/3[$, a point $a \in D(0,r)$, and let us prove that the integral $J(a) := \displaystyle\int_{D(0,\,\varepsilon)} \frac{1}{|z-a|}\left(\frac{i}{2}\right) dz \wedge d\overline{z}$ is uniformly bounded with respect to a. Indeed,

$$J(a) \;=\; \int_{D(0,\varepsilon)\setminus D(a,r/2)} \frac{1}{|z-a|}\;\left(\frac{i}{2}\right)\; dz \wedge d\overline{z} \;+\; \int_{D(a,r/2)} \frac{1}{|z-a|}\;\left(\frac{i}{2}\right)\; dz \wedge d\overline{z}$$

$\leq \frac{2}{r} \int_{D(0,\varepsilon)} \left(\frac{i}{2}\right) dz \wedge d\overline{z} \ + \int_{D(0,r/2)} \frac{1}{|z|} \left(\frac{i}{2}\right) dz \wedge d\overline{z}$
$\leq 8\pi\varepsilon.$

So, by restricting the compact set K if necessary, we can assume that for any $t \in K_1$, the roots $z^j(t)$ $(1 \leq j \leq k)$ belong to $D(0,r)$. By Fubini's theorem, we have

$$\int_K \frac{1}{|z - z^j(t)|} \left(\frac{i}{2}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_{n-1} \wedge d\overline{z}_{n-1} \wedge dz \wedge d\overline{z} \ \leq \ 8\pi\varepsilon \, \mathrm{mes}\,(K_1).$$

This completes the proof of the existence of the current $[d'f/f]$ when f is any holomorphic nonzero function on V.

§4. Proof of (LP) for "convenient "differential forms.

In this section, we prove the formula (LP) for differential forms which are locally given by $\varphi = \rho(t,z)\, dt \wedge d\overline{t}$ where ρ is a smooth function with compact support in V, and $dt \wedge d\overline{t} = dz_1 \wedge \ldots dz_{n-1} \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_{n-1}$. We will say that a such φ is "convenient "with respect to the projection $\Omega \times D(0,\varepsilon) \to \Omega$, $(t,z) \mapsto t$.

First, we consider the case where the function f is such that, for $j \neq j'$, the roots z^j and $z^{j'}$ are different at the generic point t of Ω.

By (1), and since the differential form $d'I/I$ is holomorphic on the open set $\Omega \times D(0,\varepsilon)$, we have

$$d''\left[\frac{d'f}{f}\right] \ = \ d''\left[\frac{d'P_t}{P_t}\right],$$

whence

$$d''\left[\frac{d'f}{f}\right](\varphi) \ = \ \int_{\Omega \times D(0,\varepsilon)} \frac{d'P_t}{P_t} \wedge d''\varphi$$

$= \int_{\Omega \times D(0,\varepsilon)} \left(\sum_{j=1}^{n-1} \frac{1}{P_t(z)} \ \frac{\partial P_t(z)}{\partial z_j}\, dz_j \ + \frac{P_t'(z)}{P_t(z)}\, dz \right) \wedge \frac{\partial \rho(t,z)}{\partial \overline{z}}\, d\overline{z} \wedge dt \wedge d\overline{t}$
$= \int_{\Omega \times D(0,\varepsilon)} \frac{P_t'(z)}{P_t(z)}\, dz \wedge \frac{\partial \rho(t,z)}{\partial \overline{z}}\, d\overline{z} \wedge dt \wedge d\overline{t}$

$=\int_\Omega dt \wedge d\overline{t}\ \int_{D(0,\varepsilon)} \frac{P_t'(z)}{P_t(z)}\ dz \wedge \frac{\partial\rho(t,z)}{\partial\overline{z}}\ d\overline{z}.$
By the case $n=1$ and Fubini's theorem, we get

$$\int_{D(0,\varepsilon)} \frac{1}{2i\pi}\, \frac{P_t'(z)}{P_t(z)}\ dz \wedge \frac{\partial\rho(t,z)}{\partial\overline{z}}\ d\overline{z} \ = \ [P_t = 0]\,(\rho(t,\cdot)),$$

and hence

$$d''\left[\frac{1}{2i\pi}\ \frac{d'P_t}{P_t}\right](\varphi) \ = \ \int_\Omega \left(\sum_{j=1}^{p} \rho(t, z^j(t))\right)\ dt \wedge d\overline{t}.$$

Now consider

$$R \ := \ \left\{ t \in \Omega\ ,\ \prod_{1 \le j < j' \le n} \left(z^j(t) - z^{j'}(t)\right)^2 = 0 \right\}.$$

By the hypothesis on the roots z^j, we know (see [5]) that R is a closed analytic set with empty interior, hence of Lebesgue measure equal to zero in Ω.

Put $\{f = 0\} = \{(t,z) \in \Omega \times D(0,\varepsilon)\ ,\quad P_t(z) = 0\}$. The local parametrization theorem for analytic sets exhibits the hypersurface $\{f = 0\}$ as a branched covering of degree k of Ω via the natural projection

$$\pi_0\ :\ \{f=0\}\ \rightarrow\ \Omega\ ,\ \ (t,z)\ \mapsto\ t,$$

and the branching locus is R. Then we have

$$\int_\Omega \left(\sum_{j=1}^{k} \rho(t, z^j(t))\right)\ dt \wedge d\overline{t} \ \ = \ \ \int_{\Omega\setminus R} \left(\sum_{j=1}^{k} \rho(t, z^j(t))\right) dt \wedge d\overline{t}$$

$=\int_{\{f=0\}\setminus\pi_0^{-1}(R)}\ \varphi.$
Let S be the singular locus of $\{f = 0\}$ and put $S_\varepsilon = \Big\{x \in \{f = 0\Big\}\ ,\ \ d(x,S) \le \varepsilon\}$ with $\varepsilon > 0$.
The Lelong theorem (see [6]) gives

$$\int_{\{f=0\}} \varphi \ = \ \lim_{\varepsilon\to 0}\ \int_{\{f=0\}\setminus S_\varepsilon} \varphi.$$

Since $\pi_0^{-1}(R)$ is a closed analytic space, it is of measure 0 in $\{f=0\}$, then $\pi_0^{-1}(R) \cap (\{f=0\} \setminus S_\varepsilon)$ is of measure 0 in the analytic complex manifold $\{f=0\} \setminus S_\varepsilon$.

Let χ_ε be the characteristic function of $\{f=0\} \setminus S_\varepsilon$ in the complex analytic manifold $\{f=0\} \setminus \pi_0^{-1}(R)$. We have

$$\int_{\{f=0\}\setminus \pi_0^{-1}(R)} \chi_\varepsilon \, \varphi \;=\; \int_{\{f=0\}\setminus S_\varepsilon} \varphi,$$

which implies

$$\lim_{\varepsilon\to 0} \int_{\{f=0\}\setminus \pi_0^{-1}(R)} \chi_\varepsilon \, \varphi \;=\; \int_{\{f=0\}} \varphi.$$

Moreover

$$\sum_{j=1}^{k} |\chi_\varepsilon \, \rho(t, z^j(t))| \;\leq\; \sum_{j=1}^{k} |\rho(t, z^j(t))|$$

where the term on the right is independant on ε and integrable since

$$\rho(t, z^j(t)) \; dt \wedge d\bar{t} \;=\; (\pi_0)_* \, \varphi$$

and since the differential form φ has continuous coefficients on V (see [4]) and has a compact support.

As $(\chi_\varepsilon \, \varphi)_\varepsilon$ converges almost everywhere to φ when ε tends to 0, the Lebesgue's dominated convergence theorem gives

$$\lim_{\varepsilon\to 0} \int_{\{f=0\}\setminus \pi_0^{-1}(R)} \chi_\varepsilon \, \varphi \;=\; \int_{\{f=0\}\setminus \pi_0^{-1}(R)} \varphi.$$

Then we have

$$\int_{\{f=0\}} \varphi \;=\; \int_{\{f=0\}\setminus \pi_0^{-1}(R)} \varphi \;=\; d''\Big[\frac{1}{2i\pi}\,\frac{d'f}{f}\Big](\varphi),$$

which establishes the formula (LP) when the branching locus does not coincide with Ω.

Now we are ready to prove (LP) for any holomorphic function on V.

We keep the notations of §1. We know that the vanishing locus of f in $\Omega \times D(0,\varepsilon)$ consists in that of the function $(t,z) \mapsto P(t,z)$ which is the branching covering of degree k over the open set Ω. This covering can be seen as an analytic application $P : \Omega \to \mathbf{C}^k$ where $\mathbf{C}^k$ is identified here to the set of monic polynomials of degree k with complex coefficients.

We call the ring of functions of P, and we denote by $\mathcal{O}(P)$, the quotient of the ring $\mathcal{O}(\Omega \times D(0,\varepsilon))$ of holomorphic functions on $\Omega \times D(0,\varepsilon)$, by the principal ideal generated by $\widetilde{P} : \Omega \times D(0,\varepsilon) \to \mathbf{C}$, $(t,z) \mapsto \widetilde{P}(t,z) := P_t(z)$.

The branched covering P is said to be reduced if the ring $\mathcal{O}(P)$ is reduced, that is, every nilpotent element in $\mathcal{O}(P)$ is zero. P is said to be irreducible if $\mathcal{O}(P)$ is integral.

Since P is reduced if and only if its branching locus R does not coincide with Ω (see [3]), we deduce that the formula (LP) has been proved when P is reduced. To get this result with any f, it is enough now to use the decomposition theorem (see [3]) which asserts that the branching covering P induced by f can be decomposed in a unique way in the form $P = \prod_j P_j^{n_j}$ where P_j are reduced (and irreducible) branching coverings, and n_j are positive integers.

Indeed, if f is any holomorphic function on the open set $V = \Omega \times D(0,\varepsilon)$, then the decomposition theorem allows to write $f = \prod_j f_j^{n_j}$ where the f_j are reduced and irreducible. From the result proved in §4 we deduce

$$d''\left[\frac{1}{2i\pi}\,\frac{d'f}{f}\right] = \sum_j n_j\, d''\left[\frac{1}{2i\pi}\,\frac{d'f_j}{f_j}\right] = \sum_j n_j\,[f_j = 0] = [f = 0].$$

The proof of (LP) is then complete for any $(n-1,n-1)$- form in $\mathcal{D}(V)$ locally given by $\varphi(t,z)) = \rho(t,z)\,dt \wedge d\bar{t}$.

It remains to show that the result above is still true for any compactly supported $(n-1, n-1)$- form of class C^∞ on V. It is the aim of the following section.

§3. Proof of (LP) in the general case.

We want to prove the equality

$$(0.1) \qquad d'' \left[\frac{1}{2i\pi} \frac{d'f}{f} \right] (\varphi) \;=\; \int_{\{f=0\}} \varphi$$

for any $(n-1, n-1)$-form φ in $\mathcal{D}(V)$.

By the local parametrization theorem, we may assume that

- $V = \Omega \times \mathbf{C}$ where Ω is a domain $\mathbf{C}^{n-1}$,
- $\{f = 0\}$ is a branching covering of degree k over Ω via the projection $\pi_0 : V \to \Omega$.
- there exists a compact set K in Ω such that the support of φ is contained in $K \times \mathbf{C}$.

Then, there exists a neighborhood U of 0 in $L\left(\mathbf{C}^n, \mathbf{C}^{n-1}\right)$ such that for every $u \in U$, the projection $\pi_u := \pi_0 + u$ exhibits Ω as a branching covering of a same neighborhood Ω' of K in Ω (more precisely, such that $\pi_u^{-1}(\Omega') \cap \{f = 0\} \;\to\; \Omega'$ is a branching covering of degree k).

The following lemma shows that it is sufficient to consider "sympathic" forms with respect to the given projection.

Lemma *Let* $\pi_0 : \mathbf{C}^{n+p} \to \mathbf{C}^n$ be the canonical projection. For $u \in L\left(\mathbf{C}^{n+p}, \mathbf{C}^n\right)$, set $\pi_u = \pi_0 + u$. For any couple of integers (a, b) such that $a \le n$ et $b \le n$, and for any neighborhood U of 0 in $L\left(\mathbf{C}^{n+p}, \mathbf{C}^n\right)$, we have

$$\Lambda^{a,b}\left(\mathbf{C}^{n+p}\right)^* \;=\; \sum_{u \in U} \pi_u^* \left(\Lambda^{a,b}\left(\mathbf{C}^n\right)^*\right)$$

where $\Lambda^{a,b}(E)^*$, for a complex vector space E, denotes the space of a-linear and b-antilinear alternating forms on E.

By duality and analytic extension with respect to u, this lemma is an immediate consequence of the following result.

Proposition Let $n \in \mathbf{N}^*$ and $p \in \mathbf{N}$. Let a and b be integers such that $a \leq n$ and $b \leq n$, and let $v \in \Lambda^{a,b}\left(\mathbf{C}^{n+p}\right)^*$. If for any $u \in L(\mathbf{C}^{n+p}, \mathbf{C}^n)$ we have $u_*(v) = 0$, then $v = 0$.

Proof Following [2], we establish this result by induction on p.

For $p = 0$, the result is obvious. Then we may assume $p \geq 1$.

Setting $\mathbf{C}^{n+p} = H \oplus \mathbf{C}\, e$, we have

$$\Lambda^{a,b}(\mathbf{C}^{n+p}) = \Lambda^{a,b}(H) \oplus \Lambda^{a-1,b}(H) \wedge e \oplus \Lambda^{a,b-1}(H) \wedge \overline{e} \oplus \Lambda^{a-1,b-1}(H) \wedge e \wedge \overline{e}.$$

Let $v = v_{0,0} \oplus v_{1,0} \wedge e \oplus v_{0,1} \wedge \overline{e} \oplus v_{1,1} \wedge e \wedge \overline{e}$.

– If $v_{0,0} \neq 0$, then, by induction hypothesis, there exists $f \in L(H, \mathbf{C}^n)$ such that $f_*(v_{0,0}) \neq 0$. Putting $u = f$ on H and $u(e) = 0$, we define an element of $L(C^{n+p}, \mathbf{C}^n)$ which satisfies $u_*(v) = u_*(v_{0,0}) = f_*(v_{0,0}) \neq 0$. This establishes the result in this case.

– If $v_{1,1} \neq 0$, then, by induction hypothesis (because $a - 1 \leq n - 1$ and $b - 1 \leq n - 1$), there exists $g \in L(H, \mathbf{C}^{n-1})$ such that $g_*(v_{1,1}) \neq 0$. Put $\mathbf{C}^n = \mathbf{C}^{n-1} \oplus \mathbf{C}\, \varepsilon$, and define u in $L(\mathbf{C}^{n+p}, \mathbf{C}^n)$ by $u = g \oplus 0$ on H and $u(e) = \varepsilon$. Then the component on $\Lambda^{a-1,b-1}(\mathbf{C}^{n-1}) \wedge \varepsilon \wedge \overline{\varepsilon}$ of $u_*(v)$ is $g_*(v_{1,1}) \wedge \varepsilon \wedge \overline{\varepsilon} \neq 0$, wich completes this case.

– Assume now that $v_{0,0} = v_{1,1} = 0$ and $v_{1,0} \neq 0$ (for instance). Let w be a totally decomposed vector in $\Lambda^{a-1,b}(H)^*$ such that $\langle v_{1,0}\, , w \rangle = 1$, and put

$$w = w_1 \wedge \ldots \wedge w_{a-1} \wedge \overline{t_1} \wedge \ldots \overline{t_b},$$

where the w_i and the t_j are in H^*. Since $a - 1 \leq n - 1 < n + p - 1 = \dim_{\mathbf{C}} H$, there exists a nonzero element in $\bigcap_{i=1}^{n-1} \operatorname{Ker} w_i$.

Let h^* be an element of H^* such that $\langle h^*, h \rangle = 1$. Then

$$\langle v_{1,0} \wedge h\, , w \wedge h^* \rangle = \langle v_{1,0}\, , w \rangle = 1.$$

We deduce that $v_{1,0} \wedge h$ is nonzero in $\Lambda^{a,b}(H)$. By the induction hypothesis there exists $f \in L(H, \mathbf{C}^n)$ such that $f_*(v_{1,0} \wedge h) \neq 0$.

Consider now the linear application $g : \mathbf{C}^{n+p} \to H$ defined by $g_{|H} = \mathrm{Id}_H$ and $g(e) = h$. Then, for $u = f \circ g$ we get $u_*(v_{1,0} \wedge e) \neq 0$. Moreover, if h' is close to h in H, and if h' is close to f in $L(H, \mathbf{C}^n)$, this property will remain true. We deduce

$$f'(v_{1,0}) \wedge f'(h') + f'(v_{0,1}) \wedge \overline{f'(h')} = 0$$

for at least one f' which can be assumed to be of rank n and for any h' close to h. Since f' is of maximum rank, $f'(h')$ will describe a neighborhood of $f'(h)$ when h' describes a neighborhood of h in H. Thus, we get the desired contradiction. Indeed, if A and B are elements of $\Lambda^{a-1,b}(\mathbf{C}^n)$ and $\Lambda^{a,b-1}(\mathbf{C}^n)$ respectively, and if $A \wedge v + B \wedge \overline{v} = 0$ for any v in an open subset of $\mathbf{C}^n$, then $A = B = 0$.

This completes the proof of the proposition.

Remark : In the previous lemma, it is clear that, for a given U, it is sufficient to consider a finite number of u in U .

Example : For $n = 2$ and $\lambda \in \mathbf{C}$, consider the mappings

$$\pi_\lambda : \mathbf{C}^2 \to \mathbf{C}, \quad (z_1, z_2) \mapsto z_1 + \lambda z_2.$$

We have

$$\pi_\lambda^*(dz_1 \wedge d\overline{z}_1) = dz_1 \wedge d\overline{z}_1 + \overline{\lambda}\, dz_1 \wedge d\overline{z}_2 + \lambda\, dz_2 \wedge d\overline{z}_1 + \lambda \overline{\lambda}\, dz_2 \wedge d\overline{z}_2,$$

and for λ such that $\lambda \neq \pm \overline{\lambda}$, the family

$$\left\{ \pi_\lambda^*(dz_1 \wedge d\overline{z}_1),\ \pi_{-\lambda/2}^*(dz_1 \wedge d\overline{z}_1),\ \pi_{\overline{\lambda}}^*(dz_1 \wedge d\overline{z}_1),\ \pi_{\overline{\lambda}/4}^*(dz_1 \wedge d\overline{z}_1) \right\}$$

is free, and then generates $\Lambda^{1,1}(\mathbf{C}^2)^*$.

By the lemma, and up to a finite number of projections closed to π_0, it is sufficient to consider the case $\varphi(t,z) = \rho(t,z)\, dt \wedge \bar{t}$ where $t_1, \ldots, t_{n-1}$ and z are coordinates on $\mathbf{C}^{n-1}$ and $\mathbf{C}$ respectively, $dt \wedge d\bar{t} = dt_1 \wedge \ldots dt_{n-1} \wedge d\bar{t}_1 \wedge \ldots \wedge d\bar{t}_{n-1}$, and ρ is a smooth function on $\Omega \times \mathbf{C}$ with compact support in $K \times \mathbf{C}$.

Remark Since $\log|f|$ is plurisubharmonic on V, it is locally integrable (see [7]). Then it defines a (0,0) - current on V.

Introducing the real operator

$$d^c = \frac{d' - d''}{2i\pi}$$

we have $dd^c = \frac{i}{\pi}\, d'd''$, and then we get

$$dd^c \log|f| = [f = 0]$$

which is nothing but the usual Lelong-Poincaré formula.

References

[1] Barlet D. : Le théorème d'intégration sur un ensemble analytique complexe de P. Lelong. Séminaire de géométrie analytique. Institut Elie Cartan **5**, pp. 1-11, (1982).

[2] Barlet D. : Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie. Fonctions de plusieurs variables complexes II. Lecture Notes in Mathematics **482**, pp. 1-159.

[3] Barlet D. : Convexité de l'espace des cycles. Bull. Soc. Math. France 106, pp. 373-397, (1978).

[4] Demailly J.-P. : Courants positifs et théorie de l'intersection. Gaz. Math. **53**, pp. 131-159, (1992).

[5] Gunning P., Rossi H. : Analytic functions of several complex variables. AMS Chelsea Publishing, Providence, RI, (2009).

[6] Lelong P. : Intégration sur un ensemble analytique complexe. Bull. Soc. Math. France, **85**, pp. 239-262, (1957).

[7] Lelong P. : Fonctions plurisousharmoniques et formes différentielles positives. Dunod, Paris, Gordon & Breach, New-York, (1968).

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