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A New Closed Graph Theorem on Product Spaces

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Abstract

We obtain a new version of closed graph theorem on product spaces. Fernandez's closed graph theorem for bilinear and multilinear mappings follows as a special case.

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1. Introduction

The classical closed graph theorem [1] says that, if X, Y are Banach spaces (or Fréchet spaces) and $f : X \to Y$ a linear mapping with closed graph, then f is continuous. As the closed graph theorem is a famous theorem, there have been a lot of results on it. Especially, we can find many new types of closed graph theorem recently, such as [2, 5, 6, 7, 8, 10, 11, 12].

But whether the closed graph theorem holds for mappings defined on product spaces?

In the first years, people considered the bilinear mappings defined on product spaces. P. J. Cohen [3], in 1974, gave us a negative answer for an equivalent version of the above classical closed graph theorem. However, in 1996, C. S. Fernandez [4] showed the above classical closed graph theorem holds for bilinear and multilinear mappings defined on product spaces.

In this paper, we will give another version of closed graph theorem for bi-mappings and multi-mappings on product spaces, and show that the family of bilinear (or multilinear) mappings with closed graph is just a subfamily of bi-mappings (or multi-mappings) with closed graph. Especially, from our results, the version of closed graph theorem in [11, 12] can easily be obtained and the closed graph theorem in [4] is just a special case.

2. Main results

Definition 2.1. *Let X_1, X_2 and Y be topological vector spaces. A mapping $f : X_1 \times X_2 \to Y$ is said to be a bi-mapping if for each x^1, x_n^1, $u_n^1 \in X_1$ and x^2, x_n^2, $u_n^2 \in X_2$ with $n \in \mathbf{N}$ the following (1), (2) and (3) hold:*

(1) if $f(x_n^1, 0) \to 0$ and $f(u_n^1, 0) \to 0$, then $f(x_n^1 + u_n^1, 0) \to 0$;
if $f(0, x_n^2) \to 0$ and $f(0, u_n^2) \to 0$, then $f(0, x_n^2 + u_n^2) \to 0$;

(2) if $f(x_n^1 - x^1, 0) \to 0$ and $t_n \to t$ in the scalar field $\mathbf{K}$, then

$$f(t_n x_n^1 - t x^1, 0) \to 0;$$

if $f(0, x_n^2 - x^2) \to 0$ and $t_n \to t$ in the scalar field $\mathbf{K}$, then

$$f(0, t_n x_n^2 - t x^2) \to 0;$$

(3) if $x_n^1 \to x^1$ and $x_n^2 \to x^2$, then

$$f(x_n^1, x_n^2) \to f(x^1, x^2)$$

if and only if

$$f(x_n^1 - x^1, 0) \to 0 \text{ and } f(0, x_n^2 - x^2) \to 0.$$

Note that a Fréchet space is a complete metrizable linear space. However, a Fréchet space is also a separated complete paranormed space [9].

Theorem 2.2. *Let X_1, X_2 and Y be Fréchet spaces. If $f : X_1 \times X_2 \to Y$ is a bi-mapping with closed graph, then f is continuous.*

Proof. Let $X_1 = (X_1, \|\cdot\|_1)$, $X_2 = (X_2, \|\cdot\|_2)$ and $Y = (Y, \|\cdot\|)$ where $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|$ are paranorms [9] on X_1, X_2 and Y separatively. Define a mapping

$$d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbf{R}$$

by

$$d((x_1, x_2), (u_1, u_2)) = \|x_1 - u_1\|_1 + \|x_2 - u_2\|_2 + \|f(x_1, x_2) - f(u_1, u_2)\|$$

for all x_1, $u_1 \in X_1$ and x_2, $u_2 \in X_2$. It is easy to know d is a metric on $X_1 \times X_2$.

Let $\{(x_n^1, x_n^2)\}$ be Cauchy in $(X_1 \times X_2, d)$. Then

$$d((x_n^1, x_n^2), (x_m^1, x_m^2)) = \|x_n^1 - x_m^1\|_1 + \|x_n^2 - x_m^2\|_2 + \|f(x_n^1, x_n^2) - f(x_m^1, x_m^2)\| \to 0$$

when $n, m \to +\infty$. So $\{x_n^1\}$, $\{x_n^2\}$ and $\{f(x_n^1, x_n^2)\}$ are Cauchy in $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ and $(Y, \|\cdot\|)$ respectively. Since X_1, X_2 and Y are complete, there exist $x^1 \in X_1$, $x^2 \in X_2$ and $y \in Y$ such that

$$\|x_n^1 - x^1\|_1 \to 0, \quad \|x_n^2 - x^2\|_2 \to 0, \quad \|f(x_n^1, x_n^2) - y\| \to 0.$$

But f has closed graph. Then $y = f(x^1, x^2)$ and

$$\begin{aligned} d((x_n^1, x_n^2), (x^1, x^2)) &= \|x_n^1 - x^1\|_1 + \|x_n^2 - x^2\|_2 + \|f(x_n^1, x_n^2) - f(x^1, x^2)\| \\ &= \|x_n^1 - x^1\|_1 + \|x_n^2 - x^2\|_2 + \|f(x_n^1, x_n^2) - y\| \to 0 \end{aligned}$$

when $n \to +\infty$. Hence, $(X_1 \times X_2, d)$ is a complete metric space.

Let $(x_n^1, x_n^2) \to (x^1, x^2)$ and $(u_n^1, u_n^2) \to (u^1, u^2)$ in $(X_1 \times X_2, d)$. Then

$$d((x_n^1, x_n^2), (x^1, x^2)) = \|x_n^1 - x^1\|_1 + \|x_n^2 - x^2\|_2 + \|f(x_n^1, x_n^2) - f(x^1, x^2)\| \to 0,$$

$$d((u_n^1, u_n^2), (u^1, u^2)) = \|u_n^1 - u^1\|_1 + \|u_n^2 - u^2\|_2 + \|f(u_n^1, u_n^2) - f(u^1, u^2)\| \to 0$$

when $n \to +\infty$. As f is a bi-mapping, by (3),

$$\|f(x_n^1 - x^1, 0)\| \to 0, \quad \|f(0, x_n^2 - x^2)\| \to 0,$$

and

$$\|f(u_n^1 - u^1, 0)\| \to 0, \quad \|f(0, u_n^2 - u^2)\| \to 0.$$

And by (1),

$$\|f(x_n^1 + u_n^1 - x^1 - u^1, 0)\| \to 0, \quad \|f(0, x_n^2 + u_n^2 - x^2 - u^2)\| \to 0.$$

Since

$$\|x_n^1 + u_n^1 - x^1 - u^1\|_1 \leq \|x_n^1 - x^1\|_1 + \|u_n^1 - u^1\|_1 \to 0$$

and

$$\|x_n^2 + u_n^2 - x^2 - u^2\|_2 \leq \|x_n^2 - x^2\|_2 + \|u_n^2 - u^2\|_2 \to 0,$$

by (3) again,

$$\|f(x_n^1 + u_n^1, x_n^2 + u_n^2) - f(x^1 + u^1, x^2 + u^2)\| \to 0.$$

Thus,

$$\begin{aligned} & d((x_n^1 + u_n^1, x_n^2 + u_n^2), (x^1 + u^1, x^2 + u^2)) \\ = \ & \|x_n^1 + u_n^1 - x^1 - u^1\|_1 + \|x_n^2 + u_n^2 - x^2 - u^2\|_2 \\ & + \|f(x_n^1 + u_n^1, x_n^2 + u_n^2) - f(x^1 + u^1, x^2 + u^2)\| \to \ 0 \end{aligned}$$

so the additive operation is continuous on $(X_1 \times X_2, d)$.

Let $(x_n^1, x_n^2) \to (x^1, x^2)$ in $(X_1 \times X_2, d)$ and $t_n \to t$ in the scalar field $\mathbf{K}$. Then

$$d((x_n^1, x_n^2), (x^1, x^2)) = \|x_n^1 - x^1\|_1 + \|x_n^2 - x^2\|_2 + \|f(x_n^1, x_n^2) - f(x^1, x^2)\| \to 0$$

so

$$\|x_n^1 - x^1\|_1 \to 0, \quad \|x_n^2 - x^2\|_2 \to 0, \quad \|f(x_n^1, x_n^2) - f(x^1, x^2)\| \to 0.$$

By (3), $\|f(x_n^1 - x^1, 0)\| \to 0$ and $\|f(0, x_n^2 - x^2)\| \to 0$. And by (2),

$$\|f(t_n x_n^1 - t x^1, 0)\| \to 0, \quad \|f(0, t_n x_n^2 - t x^2)\| \to 0.$$

By (3) again,

$$f(t_n x_n^1, t_n x_n^2) \to f(tx^1, tx^2)$$

since $\|t_n x_n^1 - tx^1\|_1 \to 0$ and $\|t_n x_n^2 - tx^2\|_2 \to 0$. Hence,

$$\begin{aligned} d(t_n(x_n^1, x_n^2), t(x^1, x^2)) &= \|t_n x_n^1 - tx^1\|_1 + \|t_n x_n^2 - tx^2\|_2 \\ &+ \|f(t_n x_n^1, t_n x_n^2) - f(tx^1, tx^2)\| \to 0 \end{aligned}$$

so the scalar multiplication is also continuous in $(X_1 \times X_2, d)$.

It follows that $(X_1 \times X_2, d)$ is a complete metric vector space. Namely, it is a Fréchet space. Let $I(x_1, x_2) = (x_1, x_2)$ for each $(x_1, x_2) \in X_1 \times X_2$. Then

$$I : (X_1 \times X_2, d) \to X_1 \times X_2$$

is continuous, one to one and surjective. By Banach open mapping theorem [9], the inverse

$$I^{-1} : X_1 \times X_2 \to (X_1 \times X_2, d)$$

is continuous too.

Let $\|x_n^1 - x^1\|_1 \to 0$ and $\|x_n^2 - x^2\|_2 \to 0$.

Then

$$(x_n^1, x_n^2) = I^{-1}(x_n^1, x_n^2) \to I^{-1}(x^1, x^2) = (x^1, x^2)$$

in $(X_1 \times X_2, d)$ so

$$d((x_n^1, x_n^2), (x^1, x^2)) = \|x_n^1 - x^1\|_1 + \|x_n^2 - x^2\|_2 + \|f(x_n^1, x_n^2) - f(x^1, x^2)\| \to 0.$$

Hence, $\|f(x_n^1, x_n^2) - f(x^1, x^2)\| \to 0$ so $f(x_n^1, x_n^2) \to f(x^1, x^2)$ in Y. □

Remark 2.3. *If, in Theorem 2.2, X_1, X_2, Y are Banach spaces, then every bilinear mapping f from $X_1 \times X_2$ to Y with closed graph is a bi-mapping with closed graph. So the closed graph theorem for bilinear mappings [4] is just a special case of Theorem 2.2.*

Remark 2.4. *For a weakly quasi-linear mapping f [12] from a Hausdorff topological vector space X to a topological vector space Y, define $g : X \times Z \to Y$ by $g(x, z) = f(x)$ where Z is a topological vector space. Then g is a bi-mapping from $X \times Z$ to Y, even g is with closed graph when f is with closed graph. Hence, the result of closed graph theorem in [11, 12] can immediately be obtained from Theorem 2.2.*

However, there are some bi-mappings which are not bilinear, not weakly quasi-linear, and even continuous.

Example 2.5. *Define* $f : \mathbf{R}^2 \to \mathbf{R}$ *by* $f(x, u) = x^2u^2$.

It is obvious that f *satisfies the condition (1), (2) and the necessity of (3) in Definition 2.1. Let*

$$x_n \to x, \quad u_n \to u, \quad f(x_n - x, 0) \to 0, \quad f(0, u_n - u) \to 0.$$

Then $x_n^2u_n^2 \to x^2u^2$ *in real space* $\mathbf{R}$ *by the property of product limit. So*

$$f(x_n, u_n) = x_n^2u_n^2 \to x^2u^2 = f(x, u)$$

in $\mathbf{R}$. *Hence,* f *is a bi-mapping on* $\mathbf{R}^2$.

However, f *is not bilinear, obviously.* f *is not weakly quasi-linear either. Let*

$$z_n = (x_n, u_n) = (n, \frac{1}{n^2})$$

and

$$z'_n = (x'_n, u'_n) = (n^2, \frac{1}{n^3})$$

where $n \in \mathbf{N}$. *Obviously,*

$$f(z_n) = f(x_n, u_n) = x_n^2u_n^2 = n^2 \cdot \frac{1}{n^4} \to 0$$

and

$$f(z'_n) = f(x'_n, u'_n) = (x'_n)^2(u'_n)^2 = n^4 \cdot \frac{1}{n^6} \to 0.$$

But

$$\begin{aligned} f(z_n + z'_n) &= f((x_n, u_n) + (x'_n, u'_n)) = f((x_n + x'_n), (u_n + u'_n)) \\ &= (x_n + x'_n)^2(u_n + u'_n)^2 = (x_nu_n + x_nu'_n + x'_nu_n + x'_nu'_n)^2 \\ &= \left(n \cdot \frac{1}{n^2} + n \cdot \frac{1}{n^3} + n^2 \cdot \frac{1}{n^2} + n^2 \cdot \frac{1}{n^3}\right)^2 \to 1 \neq 0 \end{aligned}$$

so f is not weakly quasi-linear.

Remark 2.6. *For* $n \geq 2$, *define* $f : \mathbf{R}^2 \to \mathbf{R}$ *by* $f(x, u) = x^nu^n$. *Then* f *is a bi-mapping, but not bilinear, not weakly quasi-linear, and even continuous.*

Example 2.7. *Define* $f : \mathbf{R}^2 \to \mathbf{R}$ *by* $f(x,u) = \sqrt{x^2+u^2}$. *Then* f *is a bi-mapping, but not bilinear.*

In the following, we will give some propositions which are helpful to our knowledge of bi-mappings in further.

As in [12], denote by $wql(X,Y)$, the family of all weakly quasi-linear mappings from the topological vector space X to the topological vector space Y.

Proposition 2.8. *Let* X_1, X_2 *and* Y *be Hausdorff topological vector spaces and* X_1, X_2 *finite-dimensional. If* $g \in wql(X_1,Y)$ *and* $h \in wql(X_2,Y)$, *and* $f : X_1 \times X_2 \to Y$ *is defined by*

$$f(x,u) = \alpha g(x) + \beta h(u),\ \forall\, x \in X_1,\ u \in X_2$$

for some $\alpha, \beta \in \mathbf{R}$, *then* f *is a continuous bi-mapping, but not bilinear for* g *or* h *is not linear.*

Proof. As in [12], we know $g(0) = 0$, $h(0) = 0$ and g, h both are continuous. So it is easy to know f is continuous and a bi-mapping for $g \in wql(X_1,Y)$ and $h \in wql(X_2,Y)$. □

Proposition 2.9. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be nontrivial paranormed space [9]. Define* $f : X_1 \times X_2 \to \mathbf{R}$ *by*

$$f(x,u) = \alpha\|x\|_1 + \beta\|u\|_2,\ \forall\, x \in X_1,\ u \in X_2.$$

Then f *is a bi-mapping but* f *is not bilinear when* $\|\cdot\|_1 \neq 0$ *or* $\|\cdot\|_2 \neq 0$.

Proof. Following the definition of paranorm [9], it is easy to know. □

Proposition 2.10. *Let* $\varphi : [0,+\infty) \to (0,+\infty)$ *and* $\psi : [0,+\infty) \to (0,+\infty)$ *be continuous functions such that*

$$0 < \mu = \inf_{t\geq 0} \varphi(t) \leq \sup_{t\geq 0} \varphi(t) = M < +\infty.$$

$$0 < \mu = \inf_{t\geq 0} \psi(t) \leq \sup_{t\geq 0} \psi(t) = M < +\infty.$$

Let $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ *be Fréchet spaces and* Y *topological vector spaces. If* $g \in wql(X_1,Y)$ *and* $h \in wql(X_2,Y)$ *are continuous, and* $f : X_1 \times X_2 \to Y$ *is defined by*

$$f(x,u) = \varphi(\|x\|_1)\, g(x) + \psi(\|u\|_2)\, h(u),\ \forall x \in X_1,\ u \in X_2,$$

then f *is a continuous bi-mapping, but not bilinear.*

Proof. We know $\|\cdot\|_1 : X_1 \to \mathbf{R}$ is continuous [9]. So $\varphi(\|\cdot\|_1)\, g(\cdot) : X_1 \to Y$ is continuous for continuous mappings φ and g. As the same, $\psi(\|\cdot\|_2)\, h(\cdot) : X_2 \to Y$ is continuous for continuous mappings ψ and h. Then f is continuous on $X_1 \times X_2$.

As in [12], $g(0) = h(0) = 0$. Since $0 < \mu \le \varphi(t) \le M < +\infty$ and $0 < \mu \le \psi(t) \le M < +\infty$ for all $t \ge 0$, $\varphi(\|x_n\|_1)\, g(x_n) \to 0$ if and only if $g(x_n) \to 0$, $\psi(\|u_n\|_2)\, h(u_n) \to 0$ if and only if $h(u_n) \to 0$. Thus, (1) and (2) hold for f for $g \in wql(X_1, Y)$, $h \in wql(X_2, Y)$.

Also, for $g(0) = h(0) = 0$, $g \in wql(X_1, Y)$ and $h \in wql(X_2, Y)$, (3) hold for f since f, g, h are continuous and φ, ψ are continous at 0. Thus, f is a bi-mapping. □

Proposition 2.11. *Let X_1, X_2 and Y be metric linear spaces and $f : X_1 \times X_2 \to Y$ a bi-mapping. If f satisfies*

$$f(x_n, 0) \to f(x, 0) \Longrightarrow f(x_n - x, 0) \to 0,$$

$$f(0, u_n) \to f(0, u) \Longrightarrow f(0, u_n - u) \to 0,$$

and for each $x \in X$ and $u \in U$, $f(\cdot, u) : X_1 \to Y$, $f(x, \cdot) : X_2 \to Y$ are continuous, then f is continuous.

Proof. If $x_n \to x$ in X_1 and $u_n \to u$ in X_2, then $f(x_n, 0) \to f(x, 0)$ and $f(0, u_n) \to f(0, u)$ so $f(x_n - x, 0) \to 0$, $f(0, u_n - u) \to 0$ and then $f(x_n, u_n) \to f(x, u)$ since f is a bi-mapping. Thus, f is continuous. □

Proposition 2.12. *Let X_1, X_2 and Y be topological vector spaces and f a mapping from $X_1 \times X_2$ to Y. If $f(\cdot, 0) \in wql(X_1, Y)$, $f(0, \cdot) \in wql(X_2, Y)$ and f is continuous, then f is a bi-mapping from $X_1 \times X_2$ to Y.*

Proof. It is easy to know (1) and (2) hold for f since $f(\cdot, 0) \in wql(X_1, Y)$, $f(0, \cdot) \in wql(X_2, Y)$.

If $x_n \to x$ in X_1 and $u_n \to u$ in X_2, then $f(x_n, u_n) \to f(x, u)$, $f(x_n, 0) \to f(x, 0)$ and $f(0, u_n) \to f(0, u)$ since f is continuous. So $f(x_n - x, 0) \to 0, f(0, u_n - u) \to 0$ since $f(\cdot, 0) \in wql(X_1, Y)$, $f(0, \cdot) \in wql(X_2, Y)$. □

Similarly, we can define multi-mappings on topological vector spaces and obtain the multi-mappings version of closed graph theorem on product spaces as follows.

Definition 2.13. *Let X_1, X_2, $\cdots$, X_m and Y be topological vector spaces. A mapping*

$$f : X_1 \times X_2 \times \cdots \times X_m \to Y$$

is said to be a multi-mapping if for each x^i, x_n^i, $u_n^i \in X_i$ *with* $i = 1, 2, \cdots, m$ *and* $n \in \mathbf{N}$ *the following (1), (2) and (3) hold:*

(1) *if* $f(0, \cdots, 0, x_n^i, 0, \cdots, 0) \to 0$ *and* $f(0, \cdots, 0, u_n^i, 0, \cdots, 0) \to 0$, *then*

$$f(0, \cdots, 0, x_n^i + u_n^i, 0, \cdots, 0) \to 0$$

where $i = 1, 2, \cdots, m$;

(2) *if* $f(0, \cdots, 0, x_n^i - x^i, 0, \cdots, 0) \to 0$ *and* $t_n \to t$ *in the scalar field* $\mathbf{K}$, *then*

$$f(0, \cdots, 0, t_n x_n^i - t x^i, 0, \cdots, 0) \to 0$$

where $i = 1, 2, \cdots, m$;

(3) *if* $x_n^i \to x^i$, $i = 1, 2, \cdots, m$, *then*

$$f(x_n^1, x_n^2, \cdots, x_n^m) \to f(x^1, x^2, \cdots, x^m)$$

if and only if

$$f(0, \cdots, 0, x_n^i - x^i, 0, \cdots, 0) \to 0 \text{ for all } i = 1, 2, \cdots, m$$

Theorem 2.14. *Let* X_1, X_2, $\cdots$, X_m *and* Y *be Fréchet spaces. If*

$$f : X_1 \times X_2 \times \cdots \times X_m \to Y$$

is a multi-mapping with closed graph, then f *is continuous.*

Remark 2.15. *It is similar to Remark 2.3, we know Fernandez's closed graph theorem on product spaces for multilinear mappings in [4] is just a special case of Theorem 2.14.*

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