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Jensen's and the quadratic functional equations with an endomorphism

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Abstract

We determine the solutions $f : S \to H$ of the generalized Jensen's functional equation

$$f(x+y) + f(x+\varphi(y)) = 2f(x), \quad x, y \in S,$$

and the solutions $f : S \to H$ of the generalized quadratic functional equation

$$f(x+y) + f(x+\varphi(y)) = 2f(x) + 2f(y), \quad x, y \in S,$$

where S is a commutative semigroup, H is an abelian group (2-torsion free in the first equation and uniquely 2-divisible in the second) and φ is an endomorphism of S.

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1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

S is a commutative semigroup [a set equipped with an associative composition rule $(x,y) \mapsto x+y$], $\varphi : S \to S$ is an endomorphism and $(H,+)$ denotes an abelian group with neutral element 0. We say that H is 2-torsion free if [h$\in H$ and $2h = 0] \Rightarrow h = 0$. H is said to be uniquely 2-divisible if for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $A : S \to H$ is said to be additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in S$.

We recall that the Cauchy difference Cf of a function $f : S \to H$ is defined by

$$Cf(x,y) := f(x+y) - f(x) - f(y), \quad x, y \in S.$$

2. Introduction

Let $\sigma \in Hom(S,S)$ satisfy $\sigma^2 = id$. In [11], Sinopoulos determined the general solution $f : S \to H$, where H is 2-torsion free, of Jensen's functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x), \quad x, y \in S, \tag{2.1}$$

and the general solution $f : S \to H$, where H is uniquely 2-divisible, of the quadratic functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.2}$$

The purpose of the present paper is to solve the functional equations (2.1) and (2.2) on semigroups without using the condition $\sigma^2 = id$. More precisely, we solve the functional equations

$$f(x+y) + f(x+\varphi(y)) = 2f(x), \quad x, y \in S, \tag{2.3}$$

$$f(x+y) + f(x+\varphi(y)) = 2f(x) + 2f(y), \quad x, y \in S, \tag{2.4}$$

where φ is an endomorphism of S. By elementary methods we show that our formulas for the solutions of (2.3) (resp. (2.4)) on semigroups are the same as those of (2.1) (resp. (2.2)), so that our results constitute a natural extension of earlier results of, e.g., [11], from Jensen's (resp. the quadratic) functional equation with an involutive automorphism to that with an endomorphism. Note that the equation (2.3) has been solved in [3] under the assumption that φ is surjective. Here we remove this restriction.

A similar functional equation that has been studied is

$$f(x+y)+f(x+\varphi(y))=2f(x)f(y), \quad x,y \in S, \tag{2.5}$$

where $f : S \to C$ is the function to determine. Eq. (2.5) was solved in a more general framework (see [3]).

3. The generalized Jensen's functional equation

In this section, we solve the functional equation (2.3) by expressing its solutions in terms of additive functions.

Lemma 3.1. *Let $f : S \to H$ be a solution of the functional equation (2.3). Then*

$$f(x+\varphi^2(y))=f(x+y) \quad \text{for all } x,y \in S. \tag{3.1}$$

Proof. Replacing y by $\varphi(y)$ in (2.3), we get that

$$f(x+\varphi(y))+f(x+\varphi^2(y))=2f(x).$$

Using this equation and (2.3), we obtain (3.1). $\square$

Lemma 3.2. *Let S be a semigroup (not necessarily abelian), $\phi : S \to S$ an endomorphism. If $f : S \to H$ and $\Phi : S \times S \to H$ satisfy*

$$f(xy)+f(\phi(y)x)=\Phi(x,y) \quad \text{for all } x,y \in S, \tag{3.2}$$

then

$$2f(xyz)=\Phi(x,yz)-\Phi(\phi(z)x,y)+\Phi(xy,z) \quad \text{for all } x,y,z \in S. \tag{3.3}$$

Proof. Making the substitutions (x, yz), $(\phi(z)x, y)$, and (xy, z) in (3.2), we get respectively

$$\begin{aligned} f(xyz)+f(\phi(yz)x) &= \Phi(x,yz), \\ f(\phi(z)xy)+f(\phi(yz)x) &= \Phi(\phi(z)x,y) \\ f(xyz)+f(\phi(z)xy)) &= \Phi(xy,z). \end{aligned}$$

Subtracting the middle identity from the sum of the other two we get (3.3). $\square$

Theorem 3.3. *Suppose that H is 2-torsion free. The general solution $f : S \to H$ of the functional equation (2.3) is $f = A + c$, where $A : S \to H$ is an additive map such that $A \circ \varphi = -A$, and where $c \in H$ is a constant.*

Proof. The method used here is closely related to and inspired by the one in [11, Proof of Theorem 2]. Assume that $f : S \to H$ is a solution of (2.3). Replacing y by $y + \varphi(y)$ in (2.3) and using Lemma 3.1, we get

$$f(x + y + \varphi(y)) = f(x). \tag{3.4}$$

Using lemma 3.2 with $\Phi(x, y) := 2f(x)$, we find after division by 2 that

$$\begin{aligned} f(x+y+z) &= f(x) - f(\varphi(z)+x) + f(x+y) \\ &= f(x) - [f(x+z) + f(\varphi(z)+x)] + f(x+z) + f(x+y) \\ &= f(x+y) + f(x+z) - f(x). \end{aligned}$$

Setting here $z = \varphi(x)$ and using (3.4), we get

$$f(y) + f(x) = f(x + \varphi(x)) + f(x + y). \tag{3.5}$$

Interchanging x and y in (3.5), we get that $f(x + \varphi(x)) = f(y + \varphi(y))$ for all $x, y \in S$. So $f(x + \varphi(x))$ is a constant, say c. By using (3.5), we infer that the function $A(x) := f(x) - c$ is additive. Substituting f into (2.3) we see that $A \circ \varphi = -A$.

The other direction of the proof is trivial to verify. □

As an immediate consequence of Theorem 3.3, we have the following result.

Corollary 3.4 (2, Theorem 3.2). *Suppose that H is 2-torsion free and let $\sigma, \tau \in Hom(S, S)$ such that $\sigma^2 = \tau^2 = id$. The general solution $f : S \to H$ of the functional equation $f(x+\sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in S$, is $f = A+c$, where $A : S \to H$ is an additive map such that $A\circ\tau = -A\circ\sigma$, and where $c \in H$ is a constant.*

Proof. The proof follows on putting $\varphi = \tau \circ \sigma$ in Theorem 3.3. □

4. The generalized quadratic functional equation

In this section, we generalize Sinopoulos's result [11, Theorem 3] on semigroups by solving the functional equation (2.4).

The following lemma lists pertinent basic properties of any solution $f : S \to H$ of (2.4).

Lemma 4.1. *Suppose that H is 2-torsion free and let $f : S \to H$ be a solution of the functional equation (2.4).*

(a) $f \circ \varphi = f$.

(b) For all $x, y, z \in S$, we have

$$\begin{aligned} f(x+y+z) &= f(x+y) + f(x+z) \\ &\quad + f(y+z) - f(x) - f(y) - f(z). \end{aligned} \tag{4.1}$$

(c) $Cf : S \times S \to H$ is a symmetric, bi-additive map satisfying

$$Cf(x, \varphi(y)) = -Cf(x, y) \quad \text{for all } x, y \in S.$$

(d) Let $A : S \to H$ be $A(x) := f(x + \varphi(x))$, $x \in S$. Then $A \circ \varphi = A$ and A is additive.

(e) $2f(x) = Cf(x, x) + A(x)$ for all $x \in S$.

Proof. (*a*) Let us first observe that $f \circ \varphi$ is a solution of (2.4). We next replace x by $\varphi(x)$ in (2.4) we find that

$$f(\varphi(x) + y) + f(\varphi(x) + \varphi(y)) = 2f(\varphi(x)) + 2f(y).$$

Adding this equation to (2.4), we get

$$\begin{aligned} &[f(x+y) + f(\varphi(x)+y)] + [f(x+\varphi(y)) + f(\varphi(x)+\varphi(y))] \\ &\quad = 2f(x) + 2f(\varphi(x)) + 4f(y). \end{aligned}$$

Using (2.4) and the fact that H is 2-torsion free, we obtain

$$[f(x) + f(y)] + [f(x) + f(\varphi(y))] = f(x) + f(\varphi(x)) + 2f(y),$$

i.e.

$$f(x) - f \circ \varphi(x) = f(y) - f \circ \varphi(y) \quad \text{for all } x, y \in S.$$

From this last equation we infer that $f - f \circ \varphi$ is a constant in H, say c. Using the fact that $f - f \circ \varphi$ is a solution of (2.4) and that H is 2-torsion free, we see that $c = 0$.

(b) Putting $\Phi(x,y) := 2f(x)+2f(y)$ in lemma 3.2, we get after division by 2 that

$$\begin{aligned}
f(x+y+z) &= f(x)+f(y+z)-f(\varphi(z)+x)-f(y)+f(x+y)+f(z) \\
&= f(x)+f(y+z)-[f(x+z)+f(\varphi(z)+x)]+f(x+z) \\
&\quad -f(y)+f(x+y)+f(z) \\
&= f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z),
\end{aligned}$$

we get (4.1).
(c) That Cf is symmetric and bi-additive follows immediately from the very definition of Cf and (4.1). Let $x, y \in S$ be arbitrary. By help of (2.4) and (a), we get that

$$\begin{aligned}
Cf(x,\varphi(y)) &= f(x+\varphi(y))-f(x)-f\circ\varphi(y) \\
&= 2f(x)+2f(y)-f(x+y)-f(x)-f\circ\varphi(y) \\
&= f(x)+f(y)-f(x+y) \\
&= -Cf(x,y).
\end{aligned}$$

(d) A is φ-even, because

$$\begin{aligned}
A(\varphi(x)) &= f(\varphi(x)+\varphi^2(x)) = 2f\circ\varphi(x+\varphi(x)) \\
&= f(x+\varphi(x)) = A(x) \quad \text{for all } x \in S.
\end{aligned}$$

Next, let $x, y \in S$ be arbitrary. By help of (4.1) and (a), we find

$$\begin{aligned}
A(x+y) &= f((x+\varphi(x))+y+\varphi(y)) \\
&= [f(x+\varphi(x)+y)+f(x+\varphi(x)+\varphi(y))]+f(y+\varphi(y)) \\
&\quad -f(x+\varphi(x))-f(y)-f\circ\varphi(y) \\
&= 2f(x+\varphi(x))+2f(y)+f(y+\varphi(y))-f(x+\varphi(x)) \\
&\quad -f(y)-f\circ\varphi(y) \\
&= f(x+\varphi(x))+f(y+\varphi(y)) \\
&= A(x)+A(y).
\end{aligned}$$

(e) Using (2.4), we obtain

$$\begin{aligned}
Cf(x,x)+A(x) &= f(x+x)+f(x+\varphi(x))-2f(x) \\
&= 4f(x)-2f(x) = 2f(x) \quad \text{for all } x \in S.
\end{aligned}$$

The second main theorem of the present paper reads as follows.

Theorem 4.2. *Suppose that H is uniquely 2-divisible. The general solution $f : S \to H$ of the functional equation (2.4) is*

$$f(x) = Q(x, x) + A(x), \quad x \in S,$$

where $Q : S \times S \to H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$, and where $A : S \to H$ is an arbitrary additive function such that $A \circ \varphi = A$.

Proof. That all solutions of (2.4) have this form is a consequence of Lemma 4.1 and the fact that H is uniquely 2-divisible. Conversely, simple computations based on the properties of Q and A, show that the indicated functions are solutions. □

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References

[1] B. Fadli, D. Zeglami and S. Kabbaj, On a Gajda's type quadratic equation on a locally compact abelian group, Indagationes Math. **26**, pp. 660-668, (2015).

[2] B. Fadli, A. Chahbi, Iz. EL-Fassi and S. Kabbaj, On Jensen's and the quadratic functional equations with involutions, Proyecciones (Antofagasta) **35** (2), pp. 213-223, (2016).

[3] B. Fadli, S. Kabbaj, Kh. Sabour and D. Zeglami, Functional equations on semigroups with an endomorphism, Acta Math. Hungar. (2016), DOI 10.1007/s10474-016-0635-9.

[4] B. Fadli, D. Zeglami and S. Kabbaj, A variant of Jensen's functional equation on semigroups, Demonstratio Math., to appear.

[5] P. de Place Friis and H. Stetkær, On the quadratic functional equation on groups, Publ. Math. Debrecen **69**, pp. 65-93, (2006).

[6] P. Kannappan, Functional equations and inequalities with applications, Springer, New York, (2009).

[7] C.T. Ng, Jensen's functional equation on groups, Aequationes Math. **39**, pp. 85-99, (1990).

[8] C.T. Ng, Jensen's functional equation on groups, II, Aequationes Math. **58**, pp. 311-320, (1999).

[9] C.T. Ng, Jensen's functional equation on groups, III, Aequationes Math. **62**, pp. 143-159, (2001).

[10] Th. M. Rassias, Inner Product Spaces and Applications, Pitman Research Notes in Mathematics Series, Addison Wesley Longman Ltd, 376, (1997).

[11] P. Sinopoulos, Functional equations on semigroups, Aequationes Math. **59**, pp. 255-261, (2000).

[12] H. Stetkær, Functional equations on abelian groups with involution, Aequationes Math. **54**, pp. 144-172, (1997).

[13] H. Stetkær, On Jensen's functional equation on groups, Aequationes Math. **66**, pp. 100-118, (2003).

[14] H. Stetkær, Functional Equations on Groups, World Scientific Publishing Co, Singapore, (2013).

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