

Proyecciones
Vol. 22, N° 2, pp. 117-125, August 2003.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172003000200002

A COMMUTATOR RIGIDITY FOR FUNCTION GROUPS AND TORELLI'S THEOREM *

RUBÉN HIDALGO

Universidad Técnica Federico Santa María, Valparaíso – Chile

Abstract

We show that a non-elementary finitely generated torsion-free function group is uniquely determined by its commutator subgroup. In this way, we obtain a generalization of the results obtained in [2], [3] and [8]. This is well related to Torelli's theorem for closed Riemann surfaces. For a general non-elementary torsion-free Kleinian group the above rigidity property still unknown.

2000 Mathematics Subject Classification. *Primary 30F40.*

Key words and phrases. *Kleinian groups, Function groups, Torelli's theorem, Hyperbolic 3-manifolds.*

*This work was partially supported by projects UTFSM 12.03.21 and Fondecyt 1030252, 1030373.

Introduction

Let us assume that S is a closed Riemann surface of genus $g \geq 2$. Associated to S we have its homology group with integer coefficients $H_1(S, \mathbf{Z}) \cong \oplus^{2g} \mathbf{Z}$ and its complex vector space of holomorphic 1-forms $H^{1,0}(S) \cong \mathbf{C}^g$. Using integration of 1-forms over cycles we may see $H_1(S, \mathbf{Z})$ inside $H^{1,0}(S)^*$, the dual space of $H^{1,0}(S)$, as a lattice. The quotient $J(S) = H^{1,0}(S)^*/H_1(S, \mathbf{Z})$ is called the Jacobian of S , which is a complex torus of dimension g admitting a principal polarization given by the intersection form. Torelli's theorem asserts that S is uniquely determined by its jacobian and its polarization. In fact, generically S is only determined by its jacobian alone. Every known proof of Torelli's theorem uses the polarization to determine S .

Given any point $p \in S$, we have a conformal embedding $j_p : S \hookrightarrow J(S)$, defined by $j_p(q) = [\int_{p_0}^q]$. A discussion of this can be found, for instance, in [1]. We can lift $j_p(S)$ by the universal covering $\pi : H^{1,0}(S)^* \rightarrow J(S)$ to obtain a Riemann surface \tilde{S} , called the homology cover of S , so that S is conformally equivalent to $j_p(S) = \tilde{S}/H_1(S, \mathbf{Z})$.

Let F be a co-compact fuchsian group, acting on the hyperbolic plane \mathbf{H} , so that $S = \mathbf{H}/F$. We have that $\tilde{S} = \mathbf{H}/[F, F]$, where $[F, F]$ is the commutator subgroup of F , that is, the subgroup of F generated by all its commutators. In particular, if we forget the polarization of $J(S)$, then we may say that Torelli's theorem is essentially "equivalent" to the following.

Theorem 1 [8]. *Let F be a torsion-free co-compact fuchsian group. Then F is uniquely determined by its commutator subgroup $[F, F]$.*

In [2], [3] we have generalized the above result to certain types of non-elementary finitely generated torsion-free Kleinian groups (Schottky type groups and B-groups). In these notes, we prove that this still holds for finitely generated torsion-free function groups (Kleinian groups with an invariant connected component). The case of torsion-free reversing Fuchsian groups of the first kind has been studied in [6] and [7].

Theorem 2. *Let G be a finitely generated non-elementary torsion free function group. Then G is uniquely determined by its commutator subgroup $[G, G]$.*

The arguments in the proof still essentially the same as the one done in the previous notes, but the difference has to do with the correct the choice of the homology and cohomology, which is not natural, in this more general situation. This is the main reason for writing this note.

Remarks.

- (1) In higher dimensions the above property fails as can be seen from an example given in [3].
- (2) If the group has no invariant components, then we have shown in [4] and [5] that for certain type of special Kleinian groups the above rigidity property still valid.

Let us recall that the homology cover of a manifold M is the one defined by the commutator subgroup of the fundamental group of M . In this way, theorem 2 can be written in the language of 3-manifolds as follows.

Theorem 3. *Let M be a hyperbolic-three manifold with a non-abelian fundamental group $\pi_1(M)$. Assume that M has a (conformal) boundary component S (induced by its hyperbolic structure) such that the natural inclusion $i : S \hookrightarrow M$ induces a surjective homomorphism $i_* : \pi_1(S) \rightarrow \pi_1(M)$. Then the hyperbolic structure of M is uniquely determined, up to isometries, by the hyperbolic structure of its homological cover.*

Proof. Let $G \cong \pi_1(M)$ be a uniformizing group of the manifold M (of isometries of the hyperbolic three space) satisfying the hypothesis of the theorem. The group G is necessarily a function group as a consequence of the surjectivity of $i_* : \pi_1(S) \rightarrow \pi_1(M)$. The result now follows from theorem 2. \square

The above results can be seen as a step into the search of a Torelli's type theorem for analytically finite Riemann surfaces and stable Riemann surfaces.

1. Some Basic Tools

To prove theorem 2 we need some basic facts from Riemann surfaces theory. First, we need the following, which is an easy consequence of proposition E.10 in chapter V of [9].

Lemma 1. *Let G be a group of Möbius transformations and $K \neq \{I\}$ be a subgroup of G .*

- (1) *If K is a normal subgroup of G and any of them is a non-elementary Kleinian group, then the other is also a Kleinian group sharing the same region of discontinuity.*

- (2) If K is a finite index subgroup of G and any of them is a Kleinian group, then the other is also a Kleinian group with the same region of discontinuity.

Let now T be an analytically finite Riemann surface, of signature (g, k) and let \bar{T} be its compactified Riemann surface of genus g . Set $H(T)$ to be the complex vector space generated by the holomorphic and antiholomorphic 1-forms on \bar{T} and the holomorphic 1-forms on T with at most simple poles at the punctures of T . The following is well known and can be found, for instance, in [1].

Lemma 2. *The dimension of $H(T)$ is given by*

$$\dim_{\mathbf{C}} H(T) = \begin{cases} 2g, & \text{if } k = 0 \\ 2g + k - 1, & \text{otherwise.} \end{cases}$$

The pairing given by

$$(w, \alpha) \in H(T) \times H_1(T, \mathbf{C}) \rightarrow \int_{\alpha} w \in \mathbf{C}$$

defines a duality between $H(T)$ and $H_1(T, \mathbf{C})$.

To write the third observation we need some facts. Let (G, Δ) be a finitely generated torsion free non-elementary function group with commutator subgroup K . Assume we have a finite index subgroup $I < G$ containing K . We have then a finite sequence of holomorphic coverings

$$r : \Delta \rightarrow \Delta/K;$$

$$t : \Delta/K \rightarrow \Delta/I;$$

$$p : \Delta/I \rightarrow \Delta/G.$$

We also have the respective coverings at the level of 3-dimensional manifolds (with boundaries):

$$R : \widehat{\Delta} \rightarrow \widehat{\Delta}/K;$$

$$T : \widehat{\Delta}/K \rightarrow \widehat{\Delta}/I;$$

$$P : \widehat{\Delta}/I \rightarrow \widehat{\Delta}/G,$$

where $\widehat{\Delta} = \Delta \cup \mathbf{H}^3$.

We may now look at the induced homomorphism at the level of first homology with complex coefficients: $H_1(P) : H_1(\widehat{\Delta}/I, \mathbf{C}) \rightarrow H_1(\widehat{\Delta}/G, \mathbf{C})$.

Since the index of I in G is finite, the covering $P : \widehat{\Delta}/I \rightarrow \widehat{\Delta}/G$ has finite degree and, in particular, $H_1(P)$ is surjective homomorphism. Moreover, we know that the kernel of these homomorphism is generated by those loops in $\widehat{\Delta}/I$ which lift to loops on $\widehat{\Delta}/K$.

We have the natural inclusions

$$\begin{aligned} i_\Delta : \Delta &\hookrightarrow \widehat{\Delta}; \\ i_K : \Delta/K &\hookrightarrow \widehat{\Delta}/K; \\ i_I : \Delta/I &\hookrightarrow \widehat{\Delta}/I; \\ i_G : \Delta/G &\hookrightarrow \widehat{\Delta}/G. \end{aligned}$$

The induced homomorphisms at the level of homology with complex coefficients are surjective. Let us denote the respective kernels by $N_\Delta = H_1(\Delta, \mathbf{C})$, $N_K = \ker H_1(i_K) < H_1(\Delta/K, \mathbf{C})$, $N_I = \ker H_1(i_I) < H_1(\Delta/I, \mathbf{C})$ and $N_G = \ker H_1(i_G) < H_1(\Delta/G, \mathbf{C})$.

We have that N_G is given by the homology classes of those loops in Δ/G which lift to loops on Δ/K , and that $H_1(r)(N_\Delta) < N_K$, $H_1(t)(N_K) < N_I$, $H_1(p)(N_I) < N_G$.

Let us consider the induced surjective homomorphism on the quotient

$$H_1(p) : H_1(\Delta/I, \mathbf{C})/N_I \rightarrow H_1(\Delta/G, \mathbf{C})/N_G,$$

which is the same as

$$H_1(P) : H_1(\widehat{\Delta}/I, \mathbf{C}) \rightarrow H_1(\widehat{\Delta}/G, \mathbf{C}),$$

whose kernel is generated by those loops on Δ/I that lift to loops on Δ/K . If we denote by N_I^\perp (respectively, N_G^\perp) the subspace of $H(\Delta/I)$ (respectively, $H(\Delta/G)$) orthogonal to N_I (respectively, N_G), then the map

$$p^* : N_G^\perp \rightarrow N_I^\perp,$$

defined by pull-back forms, is dual to the above surjective homomorphism and, in particular, it is injective. The image $P^*(N_G^\perp)$ corresponds to those forms in N_I^\perp which are invariant under the action of pull-back of each transformation in the abelian group G/I . The following is just consequence of all the above observations.

Lemma 3. *We have that*

$$\dim_{\mathbf{C}} N_G^\perp = \dim_{\mathbf{C}} H_1(\widehat{\Delta}/G, \mathbf{C}) = \dim_{\mathbf{C}} H(\Delta/G) - l_G,$$

where l_G denotes the maximal number of homologically independent simple loops in Δ/G which lift to loops on Δ/K .

2. Proof of Theorem 2

Let us consider two non-elementary finitely generated torsion free function groups (G_1, Δ_1) and (G_2, Δ_2) such that $[G_1, G_1] = [G_2, G_2] = K$. We want to prove that $G_1 = G_2$. Set $I = G_1 \cap G_2$ and $J = \langle G_1, G_2 \rangle$.

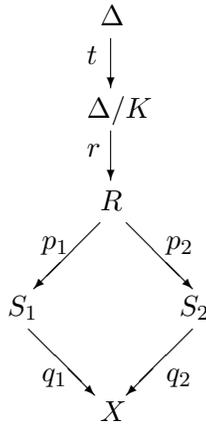
Since K is non-elementary and normal in both G_1 and G_2 , we have from lemma 1 that the region of discontinuity of the three of them is the same. Let us denote such a region by Ω . Similarly, since J is a subgroup of the normalizer of K , we have necessarily that J is a Kleinian group with Ω as region of discontinuity.

Ahlfors' finiteness theorem asserts that Ω/G_1 and Ω/G_2 are each one a finite union of finite area Riemann surfaces. It follows that the (possible branched) covering $\Omega/G_i \rightarrow \Omega/J$ has finite degree, that is, both G_1 and G_2 have finite index in J and, in particular, we have that I has finite index in G_1 , G_2 and J . It also follows from lemma 1 that I has Ω as region of discontinuity.

Claim. *There is a common invariant component Δ for G_1 and G_2 .*

Proof. If $\Delta_1 = \Delta_2$, then we consider $\Delta = \Delta_1$. Assume that $\Delta_1 \neq \Delta_2$. In this case, I keeps invariant both components Δ_1 and Δ_2 . Since I has finite index in G_i , we have that I is finitely generated. It follows from [10] that I is then a quasifuchsian group and, as a consequence, $\Omega = \Delta_1 \cup \Delta_2$. As a consequence of this, we have that Δ_1 and Δ_2 are invariant under both G_1 and G_2 . In particular, we may take $\Delta = \Delta_1$. \square

Let us consider the following notation: $\Delta/I = R$, $\Delta/G_1 = S_1$, $\Delta/G_2 = S_2$, $\Delta/J = X$. We denote by $t : \Delta \rightarrow \Delta/K$, $r : \Delta/K \rightarrow R$ and $p_i : R \rightarrow S_i$, $i = 1, 2$, the natural holomorphic coverings induced by the groups K , I/K and G_i/I , respectively. Let us consider the following commutative diagram:



We have that the induced surjective homomorphisms

$$H_1(p_1) : H_1(R, \mathbf{C})/N_I \rightarrow H_1(S_1, \mathbf{C})/N_{G_1}$$

and

$$H_1(p_2) : H_1(R, \mathbf{C})/N_I \rightarrow H_1(S_2, \mathbf{C})/N_{G_2}$$

have the same kernel: the subspace generated by the classes of those loops on R which lift to loops on Δ/K . It follows that the forms in N_I^\perp invariant under G_1/I are the same ones invariant under G_2/I . In particular, these are the same which are invariants under the group generated by G_1/I and G_2/I . It follows that the dimensions of these three spaces are the same.

By lemma 3, the dimension of the forms invariant under G_j/I (respectively, the group generated by G_1/I and G_2/I) is equal to $\dim H(S_j) - l_j$ (respectively, $\dim H(X) - l$), where l_j (respectively, l) denotes the maximal number of homologically independent simple loops on S_j that lift to loops on Δ/K (respectively, simple loops in X for which a power lift to loops on Δ/K).

In particular, we have the equalities

$$(*) \quad \dim_{\mathbf{C}} H(S_j) - l_j = \dim_{\mathbf{C}} H(X) - l.$$

Let us denote by g_j (respectively, g) the genus of S_j (respectively, X) and by k_j (respectively, k) the number of punctures of S_j (respectively, X). We have necessarily that

$$(1) \quad g \leq g_i$$

$$(2) \quad k \leq k_i$$

$$(3) \quad g - l \leq g_i - l_i$$

Inequalities (1) and (2) are clear. To obtain inequality (3), one considers a maximal set Σ of pairwise disjoint homologically independent simple loops on X all of them lifting to loops on Δ/K .

Let $\tilde{\Sigma}_i$ be the lifting of Σ to S_i . We have that $\tilde{\Sigma}_i$ is a collection of pairwise disjoint simple loops that lift to loops on Δ . We proceed to cut both X and S_i along those loops. Now glue discs along the new boundaries. We still have a (maybe branched) covering. The genus of the new surface obtained in this way from X is $g - l$. The sum of the genera from the new surfaces obtained from S_i is at most $g_i - l_i$. Now inequality (3) follows from area comparison.

The inequalities (1), (2) and (3) together with the equality (*) gives the equalities

$$g = g_i, \quad k = k_i \quad \text{and} \quad l = l_i.$$

We obtain in this way that the index of G_j in J is one and, as a consequence, $G_1 = G_2$ and we have proved theorem 2.

References

- [1] H. Farkas and I. Kra. *Riemann surfaces*. Graduate Texts in Mathematics, Springer-Verlag.
- [2] R. Hidalgo. Homology coverings of Riemann surfaces, *Tôhoku Math. J.* **45** (1993), 499-503.
- [3] R. Hidalgo, Kleinian groups with common commutator subgroup, *Complex variables* **28**, pp. 121-133, (1995).
- [4] R. Hidalgo. Noded Fuchsian groups, *Complex Variables* **36**, pp. 45-66, (1998).
- [5] R. Hidalgo. Noded function groups. *Contemporary Mathematics*. **240**, pp. 209-222, (1999).
- [6] R. Hidalgo. Homology covering of closed Klein surfaces. *Revista Proyecciones* **18**, pp. 165-173, (1999).

- [7] R. Hidalgo. A note on the homology covering of analytically finite Klein surfaces. *Complex variables* **42**, pp. 183-192, (2000).
- [8] B. Maskit. The homology covering of a Riemann surface, *Tôhoku Math. J.* **38**, pp. 561-562, (1986).
- [9] B. Maskit. *Kleinian Groups* Grundlehren der Mathematischen Wissenschaften, Vol. 287, Springer-Verlag, (1988).
- [10] B. Maskit. On boundaries of Teichmüller spaces and on kleinian groups II, *Ann. of Math.* **91**, pp. 607-639, (1970).

Received : March 2003.

Rubén Hidalgo

Departamento de Matemática
Universidad Técnica Federico Santa María
Casilla 110 - V
Valparaíso
Chile.
E-mail: rhidalgo@mat.utfsm.cl