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ASYMPTOTIC EQUILIBRIUM FOR CERTAIN TYPE OF DIFFERENTIAL EQUATIONS WITH MAXIMUM *

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Abstract

In this work we obtain asymptotic representations for the solutions of certain type of differential equations with maximum. We deduce the asymptotic equilibrium for this class of differential equations.

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1. Introduction

Differential equations with maximum arise naturally when solving practical problems, in particular, in those which appear in the study of systems with automatic regulation. A classical example is that of an electric generator. In this case, the mechanism becomes actived when the maximum voltage variation that is permited is reached in an interval of time $I_t = [t-h, t]$, with h a positive constant.The equation which describes the actioning of this regulator has the form

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t),$$

where δ and p are constants that are determined by the caracteristic of the system, $V(t)$ is the voltage and $F(t)$ is the effect of the perturbation that appears associated to the change of voltage [2].

Much work on these equations has been carried out in the last three decades. We mention the work in [1-2] and [8,11].

We study differential equations with maximum of the form

$$(1.1) \qquad \begin{cases} x'(t) = f(t, x(t), \max_{u \in I_t} x(u)), & \text{with } t \in I \\ x(t) = \varphi(t), & \text{with } t \in [-h, 0] \end{cases}$$

where $I = [0, b)$ and the possibility that b be infinity is not excluded. In addition, f is a real - valued continuous function defined on $I \times R \times R$.

We denote by $\|\varphi\|$the norm
$\|\varphi\| = \max\{|\varphi(t)| \ / \ t \in [-h, 0]\}$.

Definition 1. *A differential equation with maximum (1.1) as above has the property of asymptotic equilibrium if:*

1. *Every solution $x(t)$ of (1.1) with initial condition $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined for all $t \geq -h$ and there exists $\xi \in R$ which satisfies*

$$(1.2) \qquad \lim_{t \longrightarrow \infty} x(t) = \xi$$

2. *For all* $\xi \in R$, *there exists a solution* $x(t)$ *of (1.1), which is defined on the interval* $[-h, \infty)$ *and verifies (2.1).*

Our main result asserts, under certain hypotheses on the function f, equation (1.1) has asymptotic equilibrium. The techniques used in the proof are based on an inequality of Gronwall - Bellman type and succesive aproximations. The method used is analogous to those of [3-10].

Our principal results are applied to automatic control problems described by nonlinear equations of the type

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t, V(t)).$$

2. Main Results

In this paragraph we prove theorems about asymptotic behavior and boundedness of the solutions of equation (1.1). We first recall some basic inequality of Gronwall - Bellman type which we use in the study of the differential equation with maximum (1.1).

Lemma 1. *Let* p, x *be continuous and nonnegative functions on* $[0, b)$ *and* $\varphi_{[-h,0]}$ *a nonnegative continuous function. If the inequality*

$$x(t) \leq x(0) + \int_0^t p(s) \max_{u \in I_t} x(u)\, ds$$

holds, for all $t \in [0, b)$, *with* $x(t) = \varphi(t)$ *for all* $t \in [-h, 0]$,*then*

$$x(t) \leq \|\varphi\|\, e^{\int_0^t p(s)\, ds}$$

for all $t \in [0, b)$.

Theorem 1. *Suppose that f satisfies the following hypotheses:*

[H_0] f is continuous on $I \times R \times R$, where $I = [0, \infty)$.

[H_1] There exist λ and β integrable functions on I such that, for all $(t, x, y) \in I \times R \times R$, we have

$$|f(t, x, y)| \leq \lambda(t)(|x| + |y|) + \beta(t). \tag{2.1}$$

Then, every solution $x(t)$ with $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined on $[-h, \infty)$ and satisfies (1.2) for some $\xi \in R$. In addition, we have $x(t) = \xi + O\left(\int_t^\infty [\lambda(s) + \beta(s)]\ ds\right)$

Proof. If $x(t)$ is a solution of equation (1.1) such that $x(t) = \varphi(t)$, for all t in $[-h, 0]$, and defined on a subinterval $J = [-h, T)$ of $[-h, \infty)$, then

$$x(t) = x(0) + \int_0^t f(s, x(s), \max_{u \in I_s} x(u))\ ds, \tag{2.2}$$

for all t in $[0, T)$. Therefore, for all t in $[0, T)$, we have

$$\begin{aligned} |x(t)| &\leq |x(0)| + \int_0^t |\lambda(s)| \left(|x(s)| + \left| \max_{u \in I_s} x(u) \right| \right)\ ds + \int_0^t \beta(s) ds \\ &\leq |x(0)| + \int_0^\infty \beta(s)\, ds + \int_0^t 2\lambda(s) \max_{u \in I_s} |x(u)|\, ds. \end{aligned}$$

By lemma 1,

$$|x(t)| \leq \left[\|\varphi\| + \int_0^\infty \beta(s) ds \right] e^{\int_0^t 2\lambda(s)\ ds} \leq \left[\|\varphi\| + \int_0^\infty \beta(s) ds \right] e^{\int_0^\infty 2\lambda(s)\ ds}$$

for all t in the interval J, which shows that $x(t)$ is bounded in the interval J. Therefore, the left limit $x(T-0)$ exists, when t tends to T. Since the initial value problem

$$y'(t) = f(t, y(t), \max_{u \in I_t} y(u)),$$

with initial condition $y(t) = x(t)$ in $[T - h, T)$ and where $y(T) = x(T-0)$ has a local solution, we conclude that it is possible to extend

x beyond T. This shows that solution $x(t)$ of equation (1.1) is defined on $[-h,\infty)$ and there exists a constant M such that $|x(t)| \le M$, for all $t \in [-h,\infty)$.

Since $f(s,x(s),\max_{u\in I_s} x(u))$ is integrable on I, from equation (2.2) it follows that

$$x(t) = x(0) + \int_0^\infty f(s,x(s),\max_{u\in I_s} x(u))\, ds - \int_t^\infty f(s,x(s),\max_{u\in I_s} x(u))\, ds.$$

Let ξ denote the real number given by
x(0)+$\int_0^\infty f(s,x(s),\max_{u\in I_s} x(u))\, ds$, then, we have

$$\begin{aligned} |x(t)-\xi| &\le \int_t^\infty \lambda(s)\left(|x(s)| + \max_{u\in I_s}|x(u)|\right) ds + \int_t^\infty \beta(s)ds \\ &\le 2M \int_t^\infty \lambda(s) + \int_t^\infty \beta(s)ds \end{aligned}$$

thus,
x(t)=$\xi + O\left(\int_t^\infty [\lambda(s)+\beta(s)]\, ds\right)$.

Theorem 2. *If f satisfies conditions*

$[H_0]$ *f is continuous on $I \times R \times R$ and $f(t,0,0)$ is integrable on I, where $I = [0,\infty)$.*

$[H_2]$ *There exists a positive and integrable function μ on I such that, for all $(t,x_1,y_1),(t,x_2,y_2)$ in $I \times R \times R$, we have*

$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \mu(t)(|x_1-x_2| + |y_1-y_2|)$ *and there exist a nonnegative real constant K such that, for all $s \in [0,\infty)$*

$$\mu(s) \le K\mu(s-h).$$

Then for all $\xi \in R$ there exists a solution $x(t)$ of (1.1) which is defined on $[-h,\infty)$ and verifies (1.2). In addition, we have

$$x(t) = \xi + \int_t^\infty f(s,0,0)\ ds + O\left(\int_t^\infty \mu(s)\ ds\right) \ . \tag{2.3}$$

Proof. Let B denote the Banach space of the real bounded functions g defined on $[-h,\infty)$ with the norm defined by

$\|g\| = \{\sup|g(t)| \ / \ t \in [-h,\infty)\}$.

Consider the operator $T : B \to B$ defined by

T(x)(t)=$\xi - \int_t^\infty f(s, x(s), \max_{u\in I_s} x(u))\ ds$ for all $t \geq 0$, and by T(x)(t)=ξ, for all $t \in [-h, 0]$.

In order to prove that $T : B \to B$ has a fixed point, we must show that there exists an integer n such that T^n is a contractive operator. For this, it suffices to show that, for all x, y in B and for all positive integer n, we have

$$|(T^n y)(t) - (T^n x)(t)| \leq \frac{1}{K\ n!}\left(\int_{t-h}^\infty 2K\ \mu(s)\ ds\right)^n \|y - x\| \, . \tag{2.4}$$

We proceed by induction.

First, we verify that the inequality is true for $n = 1$.

$$\begin{aligned}
&|(Ty)(t) - (Tx)(t)| \\
= &\left|\int_t^\infty \left(f(s, y(s), \max_{u\in I_s} y(u)) - f(s, x(s), \max_{u\in I_s} x(u))\right)\ ds\right| \\
\leq &\int_t^\infty \mu(s)\left\{|y(s) - x(s)| + \max_{u\in I_s}|y(u) - x(u)|\right\}\ ds \\
\leq &\int_t^\infty 2\mu(s) \max_{u\in I_S} |y(u) - x(u)|\ ds \\
\leq &\left(\int_t^\infty 2\mu(s)\ ds\right)\|y - x\| \\
\leq &\frac{1}{K\ 1!}\left(\int_{t-h}^\infty 2K\ \mu(s)\ ds\right)\|y - x\|
\end{aligned}$$

Next, if the formula were true for $n-1$, then

$$\begin{aligned}
&|(T^n y)(t) - (T^n x)(t)| \\
&= \left| \int_t^\infty \left(f\left(s, (T^{n-1}y)(s), \max_{u\in I_s} (T^{n-1}y)(u)\right) - \right.\right. \\
&\quad \left.\left. f\left(s, (T^{n-1}x)(s), \max_{u\in I_s} (T^{n-1}x)(u)\right)\right) ds \right| \\
&\le \int_t^\infty \mu(s) \left\{ |(T^{n-1}y)(s) - (T^{n-1}x)(s)| + \right. \\
&\quad \left. \max_{u\in I_s} (T^{n-1}y)(u) - (T^{n-1}x)(u) \right\} ds \\
&\le \int_t^\infty 2\mu(s) \max_{u\in I_s} |(T^{n-1}y)(u) - (T^{n-1}x)(u)| \, ds \\
&\le \int_t^\infty 2\mu(s) \frac{1}{K(n-1)!} \left(\int_t^\infty 2K\mu(\tau)\, d\tau \right)^{n-1} \|y-x\| \, ds \\
&\le \int_t^\infty 2K\mu(s-h) \frac{1}{K(n-1)!} \left(\int_t^\infty 2K\mu(\tau)\, d\tau \right)^{n-1} \|y-x\| \, ds \\
&\le \frac{1}{K\, n!} \left(\int_t^\infty 2K\, \mu(s)\, ds \right)^n \|y-x\|.
\end{aligned}$$

Since $\mu(t)$ is integrable on I, we can find a positive integer n such that the operator T^n is a contraction. Therefore, there exists a fixed point x of T, and then x(t)=$\xi - \int_t^\infty f(s, x(s), \max_{u\in I_s} x(u))\, ds$ for all $t \ge 0$, and x(t)=ξ, for all $t \in [-h, 0]$, i. e. $x(t)$ is a solution of equation (1.1) such that $x(t) = \varphi(t)$, for all t in $[-h, 0]$.

Since $\left| x(t) - \xi - \int_t^\infty f(s,0,0)\, ds \right| \le K \int_t^\infty \mu(s)\, ds$ we can see that (2.3) is verified.

Example 1 : Consider the differential equation with maximum

$$\begin{cases} x'(t) = \frac{2}{(1+t)^2} \max\limits_{u\in I_t} x(u) \\ x(t) = e^{-2}, \text{ for } t \in [-h, 0] \end{cases}$$

We have that $f(t,x,y) = \dfrac{2}{(1+t)^2}\, y$ is a real-valued, continuous, integrable function on $I \times R \times R$ and

$$|f(t,x,y)| = \left| \frac{2}{(1+t)^2} \cdot y \right| \le \frac{2}{(1+t)^2} (|x| + |y)| \, .$$

The function $\lambda(t) = \dfrac{2}{(1+t)^2}$ is continuous and integrable on I . Thus f satisfies conditions H_0 and H_1 and by Theorems 1 and 2, every solution $x(t)$ is defined on $[-h,\infty)$ and verifies

$$x(t) = \xi + O\left(\int_t^\infty \lambda(s)ds\right) = \xi + O\left(\frac{2}{1+t}\right) \tag{2.5}$$

and reciprocally for any $\xi \in R$ there exists a solution x defined on $[-h,\infty]$ and satisfying asymptotic formula (2.5).

In this example, the function x(t)=$\begin{cases} \exp\left(-\frac{2}{1+t}\right), & \text{for} \quad t \in I \\ \exp(-2), & \text{for} \quad t \in [-h,0] \end{cases}$ is an explicit solution of the differential equation which in effect satisfies formula (2.5).

Example 2. In the differential equation with maximum

$$\begin{cases} x'(t) = \frac{2+\sin t}{1+t^2} x(t) + \frac{2}{(1+t)^2} \max\limits_{u \in I_t} x(u) \\ x(t) = \sigma \end{cases}$$

the function $f(t,x,y) = \dfrac{2+\sin t}{1+t^2}x + \dfrac{2}{(1+t)^2}y$ is continuous for $t > -1$, $x \in R$, $y \in R$.

Moreover,

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &= \mu_1(t)\,|x_1 - x_2| + \mu_2(t)\,|y_1 - y_2| \\ &\le \mu(t)\,(|x_1 - x_2| + |y_1 - y_2|) \end{aligned}$$

with $\mu(t) = \max\{\mu_1(t), \mu_2(t)\}$, where $\mu_1(t) = \dfrac{|2+\sin t|}{1+t^2}$ and $\mu_2(t) = \dfrac{2}{(1+t)^2}$.

Since f satisfies conditions H_0 , H_1 and H_2 we have that for every solution $x(t)$ of the given equation, there exists $\xi \in R$ such that x(t)=$\xi + O(\frac{\pi}{2} - \arctan t)$.

Conversely, for each $\xi \in R$, there exist a solution $x(t)$ defined on the interval $[-h,\infty)$ such that

$$x(t)=\xi+O\left(\int_t^\infty \lambda(s)ds\right)=\xi+O(\frac{\pi}{2}-\arctan t).$$

Example 3 : (Application to semilinear electrogenerators) : Consider the differential equation with maximum

$$\begin{cases} x'(t)=a(t)x(t)+g(t,x(t),\max_{s\in I_t} x(s)), \text{ for } t\in[0,\infty) \\ x(t)=e^{\int_0^t a(s)\ ds}\xi, \qquad \text{for} \qquad t\in[-h,0] \end{cases}$$

where a is a continuous function on I and g is a continuous function on $I\times R\times R$, such that f(t,x,y)=$\mathrm{e}^{-\int_0^t a(s)ds}g(t,e^{\int_0^t a(s)\ ds}x,e^{\int_0^t a(s)\ ds}y)$ verifies conditions H_0 and H_2.

Substitution $x(t)=e^{\int_0^t a(s)\ ds}v(t)$ in this equation gives
v$'(t)=e^{-\int_0^t a(s)\ ds}g(t,e^{\int_0^t a(s)\ ds}v(t),\max_{s\in I_t} e^{\int_0^s a(\tau)\ d\tau}v(s))$

By Theorem 2 , there exists a solution v of the differential equation
$\begin{cases} v'(t)=f(t,v(t),\max_{s\in I_t} v(s)) \\ v(t)=\xi\ ,\ t\in[-h,0] \end{cases}$ defined in $[-h,\infty)$ such that
v(t)=$\xi+\int_t^\infty f(s,0,0)\ ds+O\left(\int_t^\infty \mu(s)ds\right)$ then
x(t)=$\mathrm{e}^{\int_0^t a(s)\ ds}\left\{\xi+\int_t^\infty e^{-\int_0^s a(\tau)d\tau}g(s,0,0)\ ds+O\left(\int_t^\infty \mu(s)ds\right)\right\}$

If function a is integrable in the interval I then the solutions of the equation are stables. If $\int_0^t a(s)ds$ tends to $-\infty$ as t approaches infinity then, the solutions of the system are asymptotically stables.

We can apply this result to an automatic control problem described by a nonlinear equation of the type

$$\begin{cases} V'(t)=-\delta V(t)+p(t)\max_{s\in I_t} V(s)+F(t)\ ,\ \text{ if } \ t\in I \\ V(t)=e^{-\delta t}\xi\ ,\ \text{ if } \ t\in[-h,0] \end{cases}$$

where δ is a positive constant, p, F are continuous and integrable functions on I and there exists a nonnegative real constant K, such that, for all $s\in[0,\infty)$, $p(s)\le Kp(s-h)$. We can see there exists a solution

$V(t)$of this equation such that the following asymptotic representation is verified V(t)=e$^{-\delta t}\left\{\xi + \int\limits_t^\infty e^{-\delta s}F(s)\ ds + O\left(\int\limits_t^\infty p(s)ds\right)\right\}$

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